Derivation of the Breguet Range Formula for Propeller Aircraft

The Breguet range equation for a propeller-driven aircraft calculates the maximum distance an airplane can fly in steady, level flight based on aerodynamic efficiency, engine fuel consumption, and fuel weight fraction.

The Breguet Range Formula

$$\text{Range} = \frac{\eta_p}{c}\left(\frac{L}{D}\right)\ln\left(\frac{W_0}{W_1}\right) \tag{1}$$

Where:

- η_p = propulsive efficiency of the propeller
- c = specific fuel consumption (SFC, fuel mass flow rate per unit of brake/shaft power)
- L/D = lift-to-drag ratio (aerodynamic efficiency)
- W_0 = initial gross weight of the aircraft (at start of cruise, including fuel)
- W_1 = final gross weight of the aircraft (at end of cruise, without consumed fuel)

Step-by-Step Derivation

1. Basic Definitions and Flight Equilibrium

The differential change in distance dR with respect to time dt is equal to the true airspeed V:

$$\frac{dR}{dt} = V \implies dR = V\,dt \tag{2}$$

As fuel is consumed, the aircraft's total weight W decreases over time. The rate of weight loss is equal to the fuel weight flow rate $\dot{W}_f$:

$$\frac{dW}{dt} = -\dot{W}_f \implies dt = -\frac{dW}{\dot{W}_f} \tag{3}$$

For steady, level flight, the aircraft is in equilibrium. Lift equals weight ($L = W$) and thrust equals drag ($T = D$). Therefore, we can relate them as:

$$\frac{L}{D} = \frac{W}{T} \implies \frac{1}{T} = \frac{L/D}{W} \tag{4}$$

2. Power and Fuel Flow for Propeller Aircraft

For a propeller aircraft, the useful propulsive power (P_a) delivered to the air is defined by thrust times velocity:

$$P_a = T \cdot V \tag{5}$$

This propulsive power is a fraction (η_p) of the engine's brake or shaft power (P_b):

$$P_a = \eta_p \cdot P_b \implies T \cdot V = \eta_p \cdot P_b \tag{6}$$

Specific fuel consumption (c) is defined as the fuel flow rate per unit of brake power:

$$\dot{W}_f = c \cdot P_b \tag{7}$$

Substituting $P_b = \frac{T \cdot V}{\eta_p}$ into the fuel flow equation gives:

$$\dot{W}_f = c \left(\frac{T \cdot V}{\eta_p} \right) \tag{8}$$

3. Combining and Integrating Distance with Respect to Weight

Substitute the time differential dt from Equation (3) into the range differential Equation (2):

$$dR = V \left(-\frac{dW}{\dot{W}_f} \right) \tag{9}$$

Now, substitute the fuel flow equation (Equation 8) into this relationship:

$$dR = -V \left[\frac{\eta_p}{c \cdot T \cdot V} \right] dW = -\frac{\eta_p}{c} \left(\frac{1}{T} \right) dW \tag{10}$$

Using the steady flight equilibrium relation $\frac{1}{T} = \frac{L/D}{W}$ from Equation (4), the velocity V cancels out completely, leaving:

$$dR = -\frac{\eta_p}{c} \left(\frac{L}{D} \right) \frac{1}{W} dW \tag{11}$$

4. Integration over the Cruise Phase

Assuming that propulsive efficiency (η_p), specific fuel consumption (c), and the lift-to-drag ratio (L/D) remain relatively constant during the cruise phase, we integrate dR from the initial weight W_0 to the final weight W_1:

$$\text{Range} = \int_{R_0}^{R_1} dR = -\frac{\eta_p}{c} \left(\frac{L}{D} \right) \int_{W_0}^{W_1} \frac{1}{W} \, dW \tag{12}$$

We can eliminate the negative sign by inverting the integration limits:

$$\text{Range} = \frac{\eta_p}{c} \left(\frac{L}{D} \right) \int_{W_1}^{W_0} \frac{1}{W} \, dW \tag{13}$$

Evaluating the integral yields the final natural logarithm equation:

$$\text{Range} = \frac{\eta_p}{c}\left(\frac{L}{D}\right)[\ln(W_0) - \ln(W_1)] \tag{14}$$

$$\text{Range} = \frac{\eta_p}{c}\left(\frac{L}{D}\right)\ln\left(\frac{W_0}{W_1}\right) \tag{15}$$

5. Conversion to the Hydrogen Fuel-Cell Form

For a hydrogen-powered aircraft using a fuel cell, the brake power is related to the hydrogen fuel mass flow rate by

$$P_b = \eta_{fc}\dot{m}_{H_2}LHV \tag{16}$$

where η_{fc} is the fuel cell efficiency, $\dot{m}_{H_2}$ is the hydrogen mass flow rate, and LHV is the lower heating value of hydrogen.

The specific fuel consumption in the original Breguet equation is therefore

$$c = \frac{\dot{m}_{H_2}}{P_b} = \frac{1}{\eta_{fc}LHV} \tag{17}$$

Substituting this expression into the Breguet range equation gives

$$R_{H_2} = \frac{\eta_p}{c}\left(\frac{L}{D}\right)\ln\left(\frac{W_0}{W_1}\right) \tag{18}$$

$$R_{H_2} = \eta_p\eta_{fc}LHV\left(\frac{L}{D}\right)\ln\left(\frac{W_0}{W_1}\right) \tag{19}$$

Using $L/D = C_L/C_D$ and expressing aircraft weight as $W = mg$, the weight ratio becomes

$$\frac{W_0}{W_1} = \frac{m_0 g}{m_1 g} = \frac{m_0}{m_1} \tag{20}$$

Thus, the hydrogen-powered Breguet range equation can be written as

$$\boxed{R_{H_2} = \eta_{fc}\eta_p\frac{C_L}{C_D}\frac{LHV}{g}\ln\left(\frac{m_0}{m_1}\right)} \tag{21}$$

Aircraft Range and Mass Derivation

1. Setting Up the Mass Rate of Change

The fuel consumption rate is dictated by the shaft power and the fuel cell's efficiency. Converting the hourly formula to standard SI units (kg/s), the rate of change of mass is:

$$\frac{dm}{dt} = -\frac{P_{shaft}}{\eta_{fc}\cdot LHV} \tag{22}$$

The required shaft power depends on Drag (D), Velocity (V), and Propeller Efficiency (η_p):

$$P_{shaft} = \frac{D \cdot V}{\eta_p} \tag{23}$$

Velocity and Drag are dependent on instantaneous weight ($W = mg$):

$$V = \sqrt{\frac{2mg}{\rho C_L A_p}} \tag{24}$$

$$D = mg\frac{C_D}{C_L} \tag{25}$$

Substituting D and V into the $\frac{dm}{dt}$ equation gives a differential equation in terms of m:

$$\frac{dm}{dt} = -\frac{mg\frac{C_D}{C_L}\sqrt{\frac{2mg}{\rho C_L A_p}}}{\eta_p(\eta_{fc} \cdot LHV)} = -Km^{3/2} \tag{26}$$

2. Defining the System Constant (K)

Isolating all constant variables into K:

$$K = \frac{g\left(\frac{C_D}{C_L}\right)\sqrt{\frac{2g}{\rho C_L A_p}}}{\eta_p(\eta_{fc} \cdot LHV)} \tag{27}$$

Substituting the given numerical values ($\eta_{fc} = 0.4$, $LHV = 120 \times 10^6$ J/kg):

$$K = \frac{9.81\left(\frac{0.02}{0.6}\right)\sqrt{\frac{2(9.81)}{0.909(0.6)(3.025)}}}{0.9(0.4 \cdot 120 \times 10^6)} \approx 2.6103 \times 10^{-8} \text{ kg}^{-1/2}\text{s}^{-1} \tag{28}$$

3. Total Time of the Trip (t_{final})

To find the total time, we separate variables and integrate the differential equation from the start mass (m_0) to the end mass (m_1):

$$\int_{m_0}^{m_1} m^{-3/2} dm = -K \int_0^{t_{final}} dt \tag{29}$$

$$\left[-2m^{-1/2}\right]_{m_0}^{m_1} = -Kt_{final} \tag{30}$$

$$t_{final} = \frac{2}{K}\left(\frac{1}{\sqrt{m_1}} - \frac{1}{\sqrt{m_0}}\right) \tag{31}$$

Plugging in $m_0 = 186$ kg and $m_1 = 178$ kg:

$$t_{final} = \frac{2}{2.6103 \times 10^{-8}}\left(\frac{1}{\sqrt{178}} - \frac{1}{\sqrt{186}}\right) \approx 124,859 \text{ s} \tag{32}$$

4. Mass as a Function of Time ($m(t)$)

Using the indefinite integral of the separated variables and solving for m:

$$-2m(t)^{-1/2} + 2m_0^{-1/2} = -Kt \tag{33}$$

$$m(t)^{-1/2} = m_0^{-1/2} + \frac{K}{2}t \tag{34}$$

$$m(t) = \left(\frac{1}{\frac{1}{\sqrt{m_0}} + \frac{K}{2}t} \right)^2 \tag{35}$$

Inserting the numerical constants:

$$m(t) = \left(\frac{1}{0.07332 + 1.3052 \times 10^{-8} \cdot t} \right)^2 \text{ [kg]} \tag{36}$$

5. Power Required as a Function of Time ($P(t)$)

The aerodynamic Power Required (P_r) is the product of Drag and Velocity ($P_r = D \cdot V$). *Note: This aerodynamic constant (1.1276) is independent of the fuel cell's efficiency and hydrogen's LHV, so it remains unchanged.*

$$P(t) = \left(m(t) g \frac{C_D}{C_L} \right) \left(\sqrt{\frac{2m(t)g}{\rho C_L A_p}} \right) \tag{37}$$

$$P(t) = m(t)^{3/2} \left(g \frac{C_D}{C_L} \sqrt{\frac{2g}{\rho C_L A_p}} \right) \tag{38}$$

Calculating the constant bracket term gives the final power function:

$$P(t) = 1.1276 \cdot m(t)^{3/2} \text{ [W]} \tag{39}$$